MALLIAVIN CALCULUS WITH APPLICATIONS TO FINANCE

Keywords: Stochastic Calculus, Probability Theory, Finance

Authors:
ADAIR ANTONIO DA SILVA NETO [IMECC - UNICAMP][1]
DIEGO SEBASTIÁN LEDESMA (SUPERVISOR) [IMECC - UNICAMP]

---

1 Context Setting

Consider a market consisting of a riskless asset S_0 with

$$\text{riskless asset} \quad \begin{cases} dS_0(t) = \rho(t)S_0(t)\,dt \\ S_0(0) = 1 \end{cases} \tag{1}$$

and a risky asset S_1 satisfying

$$\text{risky asset} \quad \begin{cases} dS_1(t) = \mu(t)S_1(t)\,dt + \sigma(t)S_1(t)\,dW(t) \\ S_1(0) > 0 \end{cases} \tag{2}$$

where $\rho(t), \mu(t)$, and $\sigma(t) \neq 0$ are $\mathbf{F}$-adapted processes satisfying the following condition

$$\mathbf{E}\left[\int_0^T (|\rho(t)| + |\mu(t)| + \sigma^2(t))\,dt\right] < \infty$$

Let $\theta_0(t)$ and $\theta_1(t)$ denote the number of units of $S_0(t)$ and $S_1(t)$, respectively. Then the value of the portfolio $\theta = (\theta_0, \theta_1)$ is $V^\theta = \theta_0 S_0 + \theta_1 S_1$.

We also suppose that the portfolio is self-financing, i.e.,

$$dV^\theta(t) = \theta_0(t)dS_0(t) + \theta_1(t)dS_1(t) \tag{3}$$

Substituting

$$\theta_0(t) = \frac{V^\theta(t) - \theta_1(t)S_1(t)}{S_0(t)}$$

---

[1] This research was supported by the São Paulo Research Foundation (FAPESP). Process number: 2023/08064-0.

into (3) and using (1) we have

$$dV^{\theta} = \rho(t)(V^{\theta}(t) - \theta_1(t)S_1(t))dt + \theta_1(t)dS_1 \tag{4}$$

Replacing (2),

$$dV^{\theta} = [\rho(t)V^{\theta}(t) + (\mu(t) - \rho(t))\theta_1(t)S_1(t)]dt + \sigma(t)\theta_1(t)S_1(t)dW(t) \tag{5}$$

Our goal is to find a replicating (hedging) portfolio

$$V^{\theta}(T) = F, \quad \mathbf{P} - a.s. \tag{6}$$

where F is $\mathscr{F}_t$-measurable. For an European call, for example, $F = \max\{S_1 - K, 0\} = (S_1 - K)^+$.

How much do we need to invest at time $t = 0$ and which portfolio $\theta(t)$ should we use? Are V^{θ} and θ unique?

We consider $(V^{\theta}(t), \theta_1(t))$ an $\mathbf{F}$-adapted process. The equations (4) and (6) form a **backward stochastic differential equation** (BSDE). To find an explicit solution, we can change the measure and apply Clark-Ocone.

Define

$$u(t) = \frac{\mu(t) - \rho(t)}{\sigma(t)} \tag{7}$$

Using the change of measure as in the last section, we can write

$$\begin{aligned} dV^{\theta} &= [\rho(t)V^{\theta}(t) + (\mu(t) - \rho(t))\theta_1(t)S_1(t)]dt + \sigma(t)\theta_1(t)S_1(t)d\widetilde{W}(t) \\ &\quad - \sigma(t)\theta_1(t)S_1(t)\sigma^{-1}(t)(\mu(t) - \rho(t))dt \\ &= \rho(t)V^{\theta}(t)dt + \sigma(t)\theta_1(t)S_1(t)d\widetilde{W}(t) \end{aligned} \tag{8}$$

Let

$$U^{\theta}(t) = e^{-\int_0^t \rho(s)\, ds} V^{\theta}(t)$$

Then using (8),

$$dU^{\theta}(t) = e^{-\int_0^t \rho(s)\, ds} \sigma(t)\theta_1(t)S_1(t)\, d\widetilde{W}(t)$$

or, equivalently,

$$e^{-\int_0^t \rho(s)\, ds} V^{\theta}(T) = V^{\theta}(0) + \int_0^T e^{-\int_0^t \rho(s)\, ds} \sigma(t)\theta_1(t)S_1(t)\, d\widetilde{W}(t) \tag{9}$$

Let us recall the following result.

Theorem 1.1 (Clark-Ocone for $L^2(\mathbf{P})$ Under Change of Measure)**.** Suppose that $F \in L^2(\mathbf{P})$ is $\mathscr{F}_T$-measurable, and that the following conditions are met

1. $\mathbf{E}_Q[|F|] < \infty$;
2. $\mathbf{E}_Q\left[\int_0^T |D_t F|^2\, dt\right] < \infty$;
3. $\mathbf{E}_Q\left[|F| \int_0^T \left(\int_0^T D_t u(s)\, dW(s) + \int_0^T u(s) D_t u(s)\, ds\right)^2\, dt\right] < \infty$.

Then

$$F = \mathbf{E}_Q[F] + \int_0^T \mathbf{E}_Q\left[\left(D_t F - F\int_t^T D_t u(s)\, d\widetilde{W}(s)\right) \middle| \mathscr{F}_t\right] d\widetilde{W}(t)$$

where $\widetilde{W}(t)$ is a Brownian motion under the measure Q and $D_t F \in \mathscr{G}^*$ is the Hida-Malliavin derivative.

Proof. Analogous to the case for $\mathbf{D}_{1,2}$. See, e.g. [Oku10]. □

Applying the generalized Clark-Ocone formula to

$$G = e^{-\int_0^t \rho(s)\, ds} F$$

we have

$$G = \mathbf{E}_Q[G] + \int_0^T \mathbf{E}_Q\left[\left(D_t G - G\int_t^T D_t u(s)\, d\widetilde{W}(s)\right) \middle| \mathscr{F}_t\right] d\widetilde{W}(t) \tag{10}$$

Comparing (9) with (10), we have $V^\theta(0) = \mathbf{E}_Q[G]$ by uniqueness, and the replicating portfolio is given by

$$\theta_1(t) = e^{-\int_0^t \rho(s)\, ds} \sigma^{-1}(t) S_1^{-1}(t) \mathbf{E}_Q\left[\left(D_t G - G\int_t^T D_t u(s)\, d\widetilde{W}(s)\right) \middle| \mathscr{F}_t\right] \tag{11}$$

In particular, if ρ and μ are constants, and $\sigma(t) = \sigma \neq 0$, then

$$u(t) = u = \frac{\mu - \rho}{\sigma}$$

is also constant, whence $D_t u = 0$. Then the equation (11) simplifies to

$$\theta_1(t) = e^{\rho(t-T)} \sigma^{-1} S_1^{-1}(t) \mathbf{E}_Q[D_t F \mid \mathscr{F}_t] \tag{12}$$

In this presentation, we consider a digital option, which has a payoff at maturity

$$F = \mathbf{1}_{[K,\infty)}(W(T))$$

where K is the exercise price. We aim to compute the conditional expectation $\mathbf{E}_Q[D_t F \mid \mathscr{F}_t]$.

2 Necessary Results

To do that, we need the following concept.

Definition 2.1 (Donsker delta function)**.** Let $Y : \Omega \longrightarrow \mathbf{R}$, $Y \in \mathscr{G}^*$. The continuous function

$$\delta_Y(\cdot) : \mathscr{R} \longrightarrow \mathscr{G}^*$$

is the **Donsker delta function** of Y if it has the property that

$$\int_{\mathbf{R}} f(y)\delta_Y(y)\, dy = f(Y) \quad \text{a.s.}$$

for all measurable $f : \mathbf{R} \longrightarrow \mathbf{R}$ such that the integral converges.

Theorem 2.1. Suppose that

1. $\alpha : [0,T] \longrightarrow \mathbf{R}^n$ is a deterministic function such that $\|\alpha\|^2 = \int_0^T \alpha^2(s)\ ds < \infty$.
2. $\varphi : [0,T] \longrightarrow \mathbf{R}^{n\times n}$ is a deterministic function such that $\|\varphi\|^2 = \sum_{i,j=1}^n \int_0^T \varphi_{ij}^2(s)\ ds < \infty$.
3. $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ is bounded.

Define, for $t \in [0,T]$,

$$Y(t) = \int_0^t \varphi(s)\ dB(s) + \int_0^t \varphi(s)\alpha(s)\ ds$$

Then

$$f(Y(T)) = V_0 + \int_0^T u(t,\omega) \diamond (\alpha(t) + W(t))\ dt$$

where $|A| = \det A$ and A is the inverse of the covariance matrix of Y,

$$V_0 = (2\pi)^{-n/2}\sqrt{|A|}\int_{\mathbf{R}^n} f(y)\exp\left(-\frac{1}{2}y^T Ay\right)\ dy$$

and

$$u(t,\omega) = (2\pi)^{-n/2}\sqrt{|A|}\int_{\mathbf{R}^n} f(y)\exp^{\diamond}\left(-\frac{1}{2}(y-Y(t))^T \diamond A(y-Y(t))\right)\diamond((y-Y(t))^T A\varphi(t))\ dy$$

Proof. Refer to [AØU01, Theorem 4.4]. □

The next result is a simpler version of [HØ03, Lemma 3.21].

Theorem 2.2. Let $\diamond_{\mathbf{P}}$ and $\diamond_Q$ denote the Wick product for the probability measures $\mathbf{P}$ and Q respectively, and u as in the Girsanov Theorem. Then, $F \diamond_{\mathbf{P}} G = F \diamond_Q G$.

Proof. Let $F = \exp^{\diamond}\left(\int_0^\infty f(t)\ dW(t)\right)$ and $G = \exp^{\diamond}\left(\int_0^\infty g(t)\ dW(t)\right)$. Then,

$$F \diamond_{\mathbf{P}} G = \exp\left(\int_0^\infty (f(s)+g(s))\ dW(s) - \frac{1}{2}\|f+g\|^2\right)$$

Applying Girsanov to F and G,

$$F = \exp^{\diamond}\left(\int_0^\infty f(s)\ d\widetilde{W}(s) - \langle f,u\rangle_{L^2}\right), \quad \text{and} \quad G = \exp^{\diamond}\left(\int_0^\infty g(s)\ d\widetilde{W}(s) - \langle g,u\rangle_{L^2}\right)$$

Computing the product,

$$\begin{aligned} F \diamond_Q G &= \exp\left(\int_0^\infty (f(s)+g(s))\ d\widetilde{W}(s) - \frac{1}{2}\|f+g\|^2 - \langle f+g,u\rangle_{L^2}\right) \\ &= \exp\left(\int_0^\infty (f(s)+g(s))\ dW(s) - \frac{1}{2}\|f+g\|^2\right) \end{aligned}$$

Hence, $F \diamond_{\mathbf{P}} G = F \diamond_Q G$ for exponential functions. By density, we have the result. □

3 Main Result

Theorem 3.1. Suppose that ρ is constant, and $u(t)$, as defined in (7), is deterministic and satisfying $\mathbf{E}[u^2(t)] < \infty$. Then the replicating portfolio for hedging $\mathbf{1}_{[K,\infty)}(W(T))$ is

$$\theta_1(t) = e^{-\rho(T-t)}(2\pi(T-t))^{-1/2}\sigma^{-1}(t)S_1^{-1}(t)\exp\left(-\frac{(K-W(t))^2}{2(T-t)}\right) \tag{13}$$

Proof. First, notice that $F = \mathbf{1}_{[K,\infty)}(W(T)) \in L^2(\mathbf{P})$. Thus, we can use (12).

Now we compute $\mathbf{E}_Q[D_tF \mid \mathscr{F}_t]$ using the Donsker delta function by taking $f(y) = \mathbf{1}_{[K,\infty)}(y)$, and $Y(T) = W(T)$. By the Theorem 2.1,

$$\mathbf{1}_{[K,\infty)}(W(T)) = \int_K^\infty (2\pi T)^{-1/2}\exp^\diamond\left(-\frac{(y-W(T))^{\diamond 2}}{2T}\right) dy$$

By the Chain Rule for the Wick product,

$$\begin{aligned} D_t(\mathbf{1}_{[K,\infty)}(W(T))) &= \int_K^\infty (2\pi T)^{-1/2}\exp^\diamond\left(-\frac{(y-W(T))^{\diamond 2}}{2T}\right) \diamond \frac{(y-W(T))}{2T}\, dy \\ &= (2\pi T)^{-1/2}\exp^\diamond\left(-\frac{(K-W(T))^{\diamond 2}}{2T}\right) \end{aligned}$$

Denoting by $\hat{\diamond}$ the Wick product with respect to the probability measure Q, then since $\hat{\diamond} = \diamond$ (Theorem 2.2), we have

$$\begin{aligned} \mathbf{E}[D_t(\mathbf{1}_{[K,\infty)}(W(T))) \mid \mathscr{F}_t] &= \mathbf{E}_Q\left[(2\pi T)^{-1/2}\exp^{\hat{\diamond}}\left(-\frac{(K-W(T))^{\hat{\diamond}2}}{2T}\right) \middle| \mathscr{F}_t\right] \\ &= (2\pi T)^{-1/2}\mathbf{E}_Q\left[\exp^{\hat{\diamond}}\left(-\frac{(K-\widetilde{W}(T)+\int_0^T u(s)\,ds)^{\hat{\diamond}2}}{2T}\right) \middle| \mathscr{F}_t\right] \\ &= (2\pi T)^{-1/2}\exp\left(-\frac{(K-W(t))^2}{2(T-t)}\right) \end{aligned}$$

□

References

[AØU01] Knut Aase, Bernt Øksendal, and Jan Ubøe. Using the donsker delta function to compute hedging strategies. *Potential Analysis*, 14(4):351–374, 2001.

[HØ03] Yaozhong Hu and Bernt Øksendal. Fractional white noise calculus and applications to finance. *Infinite dimensional analysis, quantum probability and related topics*, 6(01):1–32, 2003.

[Oku10] Yeliz Yolcu Okur. White noise generalization of the clark-ocone formula under change of measure. *Stochastic analysis and applications*, 28(6):1106–1121, 2010.