Introduction to Combinatorial and Geometric Group Theory

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Abstract

Combinatorial and geometric group theory aim to relate algebraic properties of groups with combinatorial, geometrical and topological properties of spaces on which such groups act. In this project we studied fundamental topics of these theories such as the Bass-Serre theory of groups acting on trees, the Guba-Sapir diagram groups, the Banach-Tarski paradox and amenable groups, Gromov hyperbolic groups and automata groups. We studied tools such as the Cayley graph, 2-complexes and their fundamental group, Van Kampen diagrams for words, Gromov's word growth, Dehn's decision problems, regular languages and finite state automata.

Key words:

Groups acting on spaces, Finitely presented groups, Growth in groups

Introduction

Combinatorial and geometric group theory are interested in studying groups as objects which are always seen as symmetries of some (possibly topological) space.

The type of action yields a strong connection between the structure of this group, the topology and geometry of the space and the dynamics of the action.

There are a lot of ways to approach problems in this area, correlating properties of the group with the ones of their space of action. Since many completely different topics were studied in this project, only a small number of them will be here presented, by means of three important results in apparently unrelated areas.

Results and Discussion

Given a set G with operation *, the pair (G,*) is said to be a **group** if, for any a,b∈G, we have that a*b∈G and the operation * is associative, has an identity e and admits inverses. A subset S of G is its generator if every element a∈G can be written as a multiplication (with *) of elements of S.

A **graph** consists of a set of edges and a set of vertices. A graph is finite if it has finitely many vertices.

Given a set S, any combination of elements of S is said to be a word in the alphabet S. A set containing words formed with elements of S is a **language**.[1]

An **finite state automaton** (FSA) is a finite graph M in which there is a subset of vertices called start states, another subset of vertices called accept states and all edges are directed and labeled by elements of an associated alphabet S. M is a **deterministic automaton** if it has only one start state and no two edges leaving a vertex have the same label.[1]

The set of words accepted by a deterministic FSA is called a **regular language**.

The first result here to be presented reduces the number of necessaries properties when working with regular languages.

Theorem 1 (Anisimov): A language is regular if it is accepted by a FSA.

Given G a group generated by S finite, one can define its **Cayley graph** by the graph obtained associating each element g∈G to one vertex and connecting with one edge labeled s the vertices g and g*s, for each s∈S.

A metric space is called geodesic if there exists a shortest path between any two points on it. A geodesic metric space is **hyperbolic** if there exists $\delta \geq 0$ such that any geodesic triangle (i.e. a triangle formed with geodesic segments) has the property: a δ-neighborhood around two of its sides always contains the third one.

G is a **hyperbolic group** if its Cayley graph is a hyperbolic space. There are many properties that are granted to a group if it is hyperbolic, such as satisfying an isoperimetric linear inequality. That makes the following result not only interesting but also convenient in some cases.

Theorem 2: Almost every group is hyperbolic.

Although hyperbolic groups and automata seen to be unrelated areas, there is a simple example of problem in which both are quite useful and are correlated by a property: the case of Dehn's word problem.

Dehn's word problem consists in discovering, given a∈S alphabet, if a≡1 in G the group generated by S. Every hyperbolic group has this problem solvable and with automata, one can define the set WP={ω word in the alphabet S | ω≡1 in G generated by S}. WP clearly is a language, and therefore has associated automaton by means of which it can be studied.

A group is called **paradoxical** if there are two different families of pairwise disjoint subsets of G such that G is the union of the subsets for each family.[2]

A group is called **amenable** if there exists μ an invariant mean for G, that is, a function which assign a nonnegative number to each subset of G such that μ(G)=1, μ(A∪B)= μ(A)+ μ(B) and ∀ g∈G μ(gA)= μ(A).[2]

Theorem 3 (Tarski): A group is amenable if it is non-paradoxical.

These definitions are intimately correlated to the Banach-Tarski paradox and the mathematical proof of the possibility of cutting a chicken in finite pieces and rearranging them into a dog (or just doubling a sphere).[2]

Conclusions

The student has learned many advanced concepts and tools for the study of combinatorial and geometric group theories. With this knowledge, it was possible to study important results in different topics, such as automata, hyperbolic spaces and the Banach-Tarski paradox.

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[1]Meier, J; Groups, graphs and trees: An introduction to the geometry of infinite groups; Cambridge University Press, 2008

[2]Sapir, M.; Combinatorial algebra: syntax and semantics; Springer Monographs in Mathematics; 2014