



A study on 3-Flow Theory

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Abstract

An *m3-orientation* or, equivalently, a *3-flow* is an assignment of directions and unit weights to the edges of an undirected graph, such that each vertex of G has a net outflow equal to zero modulo 3. This work deals with essentially 4-edge-connected graphs having four vertices of degree three and the remaining vertices of even degree. We characterize the graphs in this class which admit an *m3-orientation*.

Key words:

Nowhere-zero flows, *m3-orientations*, *3-Flow Conjecture*.

Introduction

A graph G is an ordered triple $(V(G), E(G), I_G)$, consisting of a nonempty set of vertices $V(G)$, a set of edges $E(G)$ disjoint from $V(G)$, and an incidence function I_G that associates each element of $E(G)$ with a pair of vertices, not necessarily distinct.

An *integer flow* for G is an ordered pair (D, f) , such that D is an orientation of the edges of G (flows) and $f : E(G) \rightarrow \mathbb{Z}^*$ is an assignment of weights to the edges of G , such that the net outflow on each vertex is zero. A *k-flow* is an integer flow for which $f : E(G) \rightarrow \{1, 2, \dots, (k-1)\}$. A *modular k-flow (mod k-flow)* is a *k-flow* for which the net outflow on each vertex is zero modulo k . A graph admits a *k-flow* if and only if it admits a *mod k-flow*. An *m3-orientation* is a *mod 3-flow* for which all the weights are equal to one. A *mod 3-flow* can be converted to an *m3-orientation* by reversing the orientation of weight-two edges and changing their weights to one.

For a proper subset X of $V(G)$, an *edge cut* $\nabla(X)$ is the set of edges with one end in X and the other in $V(G) \setminus X$. If $|X| = 1$ or $|V(G) \setminus X| = 1$, $\nabla(X)$ is a *trivial cut*. A graph G is said to be *k-edge-connected* if $|\nabla(X)| \geq k$ for each subset X of G . Moreover, such a graph G is *essentially (k+1)-edge-connected* if all of its edge cuts with cardinality k are trivial.

W. T. Tutte introduced the theory of Integer Flows³ in 1954 and posed two conjectures, known as the 4- and 5-Flow Conjectures. Later, he posed the 3-Flow Conjecture², which states that every 4-edge-connected graph admits a 3-flow. This is the subject of this work.

Results and Discussion

Let G be a graph having four vertices of degree three, u_1, u_2, u_3, u_4 , and the remaining of even degree. Let $\nabla(X)$ be an edge cut of G such that: $|\nabla(X)| = 4$; $\{u_1, u_2, u_3, u_4\}$ is a subset of X ; and, for every vertex v of X , $d(v) \leq 4$. If $G[\{u_1, u_2, u_3, u_4\}]$ is isomorphic to $K_{1,3}$, then we say that G has a *forbidden configuration*.

Theorem 1. Let G be an essentially 4-edge-connected graph, not isomorphic to K_4 , having four vertices of degree three and the remaining vertices of even degree. Then, G has an *m3-orientation* if and only if it does not have a forbidden configuration.

The proof of the theorem is based on finding six edge-disjoint paths connecting one pair of vertices of degree three to the other pair, and orienting their edges from one pair to the other. Then, the removal of these paths leaves an eulerian graph, that is known to have an *m3-orientation*. The union of these two orientations is an *m3-orientation* for the original graph.

Conclusions

One possible next step for us to pursue would be to consider a graph G with four vertices of degree three, two vertices of degree five, and all the remaining vertices of even degree. This would lead to a theory where the removal of paths connecting vertices of degree three does not necessarily produce an eulerian graph as a result but, potentially, a graph with new vertices of degree three. This could, perhaps, be part of a general theory for solving the 3-Flow Conjecture.

Acknowledgement

The student would like to thank the National Council for Scientific and Technological Development, CNPq, for the financial support.

¹ Asghar, N. *Grötzsch's Theorem*. Master's thesis, University of Waterloo, Ontario, Canada, 2012.

² Steinberg, R. *Grötzsch's Theorem Dualized*. Master's thesis, University of Waterloo, Ontario, Canada, 1976.

³ Tutte, W. T. A contribution to the theory of chromatic polynomials. *Canadian J. Math.* 6 (1954), 80-91.

⁴ Younger, D. H. Integer flows. *Journal of Graph Theory*, 7 (1983), 349-357.