An example of classical variety in projective geometry.

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Abstract

We show that given three pairwise skew lines L, M and N, in the projective space P³, there is only one Segre variety that contains these lines.

Key words:

Algebraic Geometry, Algebra, Projective Geometry

Introduction

The Segre variety is defined as the image of the map σ, given by

$$\sigma: \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$$

$$\sigma\big((X_0, \dots, X_n), (Y_0, \dots, Y_m)\big) = (\dots, X_i Y_j, \dots),$$

where the coordinates on the image range over all the pairs of coordinates X_i and Y_j. We denote the Segre variety by $\Sigma_{1,1}$.

Our goal is to find a variety that contains the three lines, and is also isomorphic to the Segre variety $\Sigma_{1,1}$.

Results and Discussion

We define the variety X as the union of all lines meeting L, M and N. We will show that there is a regular map $\varphi: X \twoheadrightarrow \Sigma_{1,1}$, such that the inverse map of φ is also regular.

We can suppose, by a change of coordinates, that $L = \langle e_1, e_2 \rangle, M = \langle e_2, e_3 \rangle$ and $N = \langle v_1, v_2 \rangle$. There exist 2×2 matrices M_1 and M_2, such that the block matrix

$$A = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$$

preserves L and M, and transforms N in $\langle e_1 + e_3, e_2 + e_4 \rangle$.

We now consider the map ϕ, defined by

$$\phi\big((\lambda:\mu), (x_1:x_2)\big) = \lambda(x_0:x_1:x_0:x_1) + \mu(x_0:x_1:0:0).$$

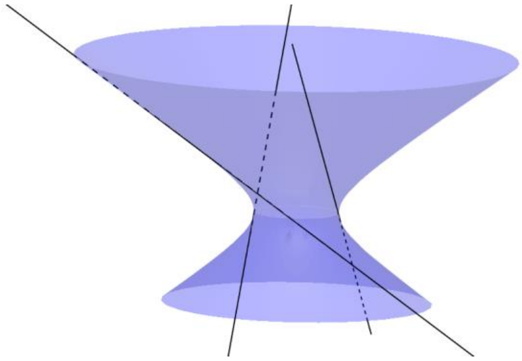
We observe that for each fixed point $z_0 = (x_0:y_0) \in \mathbb{P}^1$, the map $\phi(z_0)$ is a line in $\mathbb{P}^3$ that meet L, M and N, moreover, all the lines that meets L, M and N can be described in this way, then $Im(\phi) = X$.

Lastly, we define the regular map φ as

$$\varphi: X \to \Sigma_{1,1}$$

$$\varphi\big(\lambda(x_0:x_1:x_0:x_1) + \mu(x_0:x_1:0:0)\big) = \lambda(0:0:x_0:x_1) + \mu(x_0:x_1:0:0)$$

Figure 1. $\Sigma_{1,1}$ and the lines L, M and N.



Conclusions

Once it is shown that the inverse of the map φ is regular, we get that, indeed, X is a variety, and because the map φ is sobrejective, we conclude that $X \cong \Sigma_{1,1}$.

The uniqueness is obtained showing that the three lines impose conditions over the coefficients from the quadratic polynomials associated to the Segre variety, in order to find one and only one variety.

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[1] Joe Harris, Algebraic Geometry A First Course, Graduate Texts in Mathematics, 133, Springer Verlag – New York (1992).