On the solution of the pendulum equation via the Weierstraß ℘ function

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Abstract
We show how the mathematical pendulum gives rise to a doubly periodic (also known as elliptic) function, the Weierstraß ℘ function, which is naturally defined on a Riemann surface. Using this construction, we solve the pendulum equation without approximations.

Key words: Riemann Surfaces, Elliptic Functions, Classical Mechanics.

Introduction

In suitable units, the pendulum equation is given by $\dot{\theta}^2 - \cos\theta = E$, where E is constant. Reworking it and using the substitution $u = \cos\theta$, we find that

$$t = -\int \frac{du}{\sqrt{(E+u)(1-u^2)}},$$

an elliptic integral. We are thus interested in functions arising from solving integrals of the type

$$\int \frac{du}{\sqrt{f(u)}},$$

where f is a polynomial of degree 3 or higher, since otherwise they are known to be integrable in terms of elementary functions.

Results and Discussion

Given a non-degenerate lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$, we define the Weierstraß ℘ function as

$$\wp(u) = \frac{1}{u^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left[\frac{1}{(u-\omega^2)} - \frac{1}{\omega^2}\right].$$

This function is doubly periodic (with periods being $\mathbb{Z}$-linear combinations of ω_1 and ω_2) and meromorphic. A doubly periodic meromorphic function on $\mathbb{C}$ gives rise to a meromorphic function on the complex torus $\mathbb{C}/\Lambda$, which is a Riemann surface.

It can be shown that

$$\wp'(u) = 4\wp(u)^3 - g_2\wp(u) - g_3,$$

where g_2 and g_3 depend on Λ. This tells us that $\mathbb{C}/\Lambda$ arises as the Riemann surface associated to an equation $w^2 = f(z)$ for cubic f. This is true for *any* such f, as a convenient choice of ω_1 and ω_2 produces any preassigned values of g_2 and g_3, and the quadratic term missing from the polynomial on the right hand side of the equation above can be recovered by a change of variables.

Conclusions

Using the differential equation for ℘ given above, we can see that, for a suitable lattice Λ, the amplitude of the pendulum is given by

$$\theta = \cos^{-1}\big(\wp_\Lambda(t+t_0)\big).$$

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