XXIII Congresso de Iniciação Científica da Unicamp 2015
17 a 19 de Novembro
Campinas Brasil

The Gauss-Bonnet Theorem and Homology Groups

Rafael Leão, Ian D. de Paula

Abstract

We develop the theory of homology groups, and from them deduce the topological invariance of the Euler characteristic, a key concept to the understanding and extension of the classical Gauss-Bonnet theorem of differential geometry.

Key words: Differential Geometry, Homology, Homotopy

Introduction

In differential geometry, surfaces are studied by means of the curves on them. In this way, we naturally encounter the concept of sectional curvatures on points of a surface, related to the accelerations of the curves through that point.

We then define the Gaussian curvature as the product of the extremes of the sectional curvatures at a point. Since it is defined at points, this is a local property. Its importance is dues to Gauss's Theorema Egregium, which tells us it is an intrinsic property of the surface, i.e., it does not depend on the way the surface is immersed in a larger space.

The Gauss-Bonnet theorem is a classical result in differential geometry, which relates the Gaussian curvature of a surface to its Euler characteristic, defined for polyhedra in high school and extended to surfaces by means of polygonal approximation.

Its worth lies in that Gaussian curvature is a local geometric property, which depends heavily on a surface's shape, while the Euler characteristic is a global topological property. This allows us to conclude, for example, that the torus does not admit any metric in which its Gaussian curvature is always positive, and that it is not possible to project the Earth without distortion.

The development of this theory, however, requires that the underlying concept of topological property be precisely understood. For this purpose, we must delve into the concepts of homotopy, the continuous deformation of a mapping into another, and of homotopy equivalence, maps with compositions homotopic to identities. We can them define topological properties, or invariants, as those properties preserved by homotopy equivalences.

Our work consists in studying these concepts and then proceeding to prove that the Euler characteristic is in fact a topological invariant.

Results and Discussion

In order to show the topological invariance of the Euler characteristic, we resort to the concept of homology groups, a potent tool of algebraic topology which relates to the formation of cycles on the surface.

Conclusions

We proved the topological invariance of the Euler Characteristic, paving the way for the construction of the Gauss-Bonnet theorem, in surfaces as well as in more general geometric objects, such as smooth manifolds.

Acknowledgement

We would like to thank IMECC – Unicamp for the facilities and resources, vital to research, of which we made use.

---

[1] Hatcher, A. E. *Algebraic Topology. Cambridge University Press.* **2002.**

[2] O'Neill, B. *JElementary Differential Geometry. 2nd Ed. Academic Press.* **2006.**

[3] Carmo, M. P. *Geometria Diferencial de Curvas e Superfícies. Sociedade Brasileira de Matemática.* **2012.**

[4] Armstrong, M. A. *Basic Topology. Springer.* **1983.**

[5] Tu, L. W. *An Introduction to Manifolds. Springer.* **2008.**

[6] Lee, J. M. *Introduction to Smooth Manifolds. 2nd Ed. Springer.* **2012.**