About the existence of conformal metrics on a closed Riemannian manifold (M,g) in a such way that a closed p-form ω is harmonic.

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Abstract

We present a sufficient condition for the existence of a conformal metric h on a closed Riemannian manifold (M,g) with a closed p-form ω in a such way that $\Delta_h\omega = 0$.

Key words: Hodge theory, Laplace operator, harmonic forms.

Introduction

By Hodge theory (1930), for a closed p-form ω be harmonic in a closed Riemannian manifold (M,g) is necessary and sufficient that ω be co-closed.

$$\Delta_g\omega = 0 \Leftrightarrow d\omega = \delta\omega = 0.$$

We know, however, that δ = $*_g\, d\, *_g$ and the condition "be co-closed" depends on the metric. This allows ask:

Given a closed p-form ω on a closed Riemannian manifold (M,g), when does there exist a metric h on M in a such way that $\Delta_h\,\omega = 0$?

This question was answered completely for 1-forms by Calabi in 1969.

We present a sufficient condition for the solution in the case of p-forms in each open of some atlas over M.

Results and Discussion

We search for a metric

$$h = e^{\frac{2f}{n}} g$$

where $f : M \to \mathbb{R}$ is a C^∞function.

Let $*_g$ the Hodge star on the metric g. For ω be co-closed on h-metric:

$$d *_h \omega = 0.$$

Then the condition to ω be co-closed is the existence of a function $f \in C^\infty(M)$ such that

$$df \wedge *_g\, \omega = -d *_g \omega \quad (1)$$

We rewrite it in terms of a matrix equation:

Using the multi-index notation, any p-form ω can be written as:

$$\omega = \sum_I \omega_I dx^I,$$
$$I = \{i_1 < \cdots < i_p\}.$$

Let J be the multi-index complementary of I.

Then we define the vector S^J of $\mathbb{R}^n$ whose coordinates $j = 1, \dots, n$ are:

$$(S^J)_j := \begin{cases} (-1)^\alpha \omega_{(J-i_k)}, & j=i_k \\ 0, & j \text{ not in } J. \end{cases}$$

where α means the signal of permutation to rearrange J as an ascending multi-index.

Let A the matrix whose entries are S^J, and choose a column vector c whose components are $-\nabla . S^J$.

Then we rewrite the PDE (1) as:

$$\nabla f^t . A = c \quad (2)$$

Conclusions

Once the problem reduces to linear algebra, the sufficient condition for the existence of conformal metric is given by the invertibility of A.

But since A in invertible we get:

$$\nabla f^t = c . A^{-1}$$

Then the solution f is obtained by integration in each open set of an atlas over M.

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[1] Alcibiades Rigas, Sandro B. Cristiano, Francesco Mercuri. Differential Forms and their Integrals..